THE STATE OF STRESS IN A THIN PLATE

(NAPRIAZHENNOE SOSTOIANIE PLITY MALOI TOLSHCHINY)

PMM Vol.27, No.6, 1963, pp.1057-1074

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(Received June 24, 1963)

The problem of the theory of elasticity for a plate with prescribed stresses at the boundary will be treated. A study will be made of the behavior of the state of stress when the thickness of the plate is decreased.

Methods of constructing asymptotic processes for this problem were proposed earlier by A.L. Gol'denveizer in the lecture to the First National Congress on Theoretical and Applied Mechanics (1 Vsesoiuznyi S''ezd po Teoreticheskoi i Prikladnoi Mekhanike) in 1960, and also in the works of Friedrichs and Dressler [1], Green [2], Reiss [3] and Gol'denveizer [4].

The method presented here leads to the construction of asymptotic expressions for the step-by-step solution of a series of biharmonic problems, which are equivalent to the problem of the technical bending theory of plates, and to the inversion of a certain infinite matrix. This matrix does not depend on the geometry of the plate, and its inversion need only be carried out once for all plates and loadings.

1. We will consider a plate of isotropic, homogeneous material of thickness $2h$ (Fig. 1). The boundary of the plate consists of two planes Γ_1 and the cylindrical surface Γ_2. We will assume that the boundary Γ_1 is free of loading, and that the loading applied to Γ_2 is self-equilibrating. The assumption that Γ_1 is free has been introduced for simplicity. These stresses can always be removed by means of the solution of the corresponding problem for an infinite layer (see, for

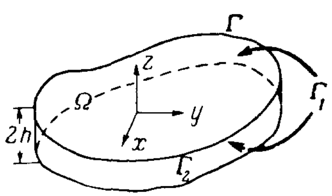

Fig. 1.

example, [5]). Moreover, for these solutions it is possible to construct asymptotic expansions for small values of h. The behavior of the state of stress caused by tractions on Γ_2 proves to be more complicated. This question forms the essential contents of the present article.

Here, the authors consider only the algorithm for the construction of the asymptotic representations, leaving aside the question of the exploitation of the said algorithm and its motivation. The whole treatment here has been carried out for the case of the flexure of a plate, although it is clear that the following presentation of the method is also applicable to the general case of the deformation of a plate.

We will start with the relations given by Lur'e [6]. It has been proved there that in the absence of loading on Γ_1 the state of stress in the plate during bending can be formed from three states of stress which henceforth will be called biharmonic, rotational and potential. The sense of these terms will become clear below.

The biharmonic state of stress is given by the following formulas:

$$u^{(1)} = \lambda a\left[(\nu + 1)\,\zeta \frac{\partial\psi}{\partial\xi} - \left(\nu + \frac{1}{3}\right)\frac{\zeta^3}{2}\lambda^2\frac{\partial\Delta\psi}{\partial\xi}\right]$$
$$v^{(1)} = \lambda a\left[(\nu + 1)\,\zeta \frac{\partial\psi}{\partial\eta} - \left(\nu + \frac{1}{3}\right)\frac{\zeta^3}{2}\lambda^2\frac{\partial\Delta\psi}{\partial\eta}\right] \qquad (1.1)$$
$$w^{(1)} = -(\nu + 1)\,a\psi - (\nu - 1)\frac{\zeta^2}{2}a\lambda^2\Delta\psi + 2\nu a\lambda^2\Delta\psi$$

$$\tau_{xz}^{(1)} = 2\mu\nu\lambda^2(1 - \zeta^2)\frac{\partial\Delta\psi}{\partial\xi}, \qquad \tau_{yz}^{(1)} = 2\mu\nu\lambda^2(1 - \zeta^2)\frac{\partial\Delta\psi}{\partial\eta}, \qquad \sigma_z^{(1)} = 0$$
$$\sigma_x^{(1)} = 2\mu\lambda\left\{\left[2\nu\frac{\partial^2\psi}{\partial\xi^2} + (\nu - 1)\frac{\partial^2\psi}{\partial\eta^2}\right]\zeta - \left(\nu + \frac{1}{3}\right)\frac{\zeta^3}{2}\lambda^2\frac{\partial^2\Delta\psi}{\partial\xi^2}\right\}$$
$$\sigma_y^{(1)} = 2\mu\lambda\left\{\left[2\nu\frac{\partial^2\psi}{\partial\eta^2} + (\nu - 1)\frac{\partial^2\psi}{\partial\xi^2}\right]\zeta - \left(\nu + \frac{1}{3}\right)\frac{\zeta^3}{2}\lambda^2\frac{\partial^2\Delta\psi}{\partial\eta^2}\right\} \qquad (1.2)$$
$$\tau_{xy}^{(1)} = 2\mu\lambda\left\{(\nu + 1)\,\zeta\frac{\partial^2\psi}{\partial\xi\partial\eta} - \left(\nu + \frac{1}{3}\right)\frac{\zeta^3}{2}\lambda^2\frac{\partial^2\Delta\psi}{\partial\xi\partial\eta}\right\}$$
$$(\zeta = z/h,\ \xi = x/a,\ \eta = y/a,\ \lambda = h/a)$$

where a is a characteristic dimension of the plate in the x, y-plane; ψ is a certain biharmonic function of the variables ξ, η; μ is the shear modulus; σ is Poisson's ratio; and Δ is the Laplace operator.

The rotational state of stress is given by the relations

$$u^{(2)} = 2\lambda^2 a\sum_{k=0}^{\infty}\frac{\sin\sigma_k\zeta}{\sigma_k}\frac{\partial B_k}{\partial\eta}, \qquad v^{(2)} = -2\lambda^2 a\sum_{k=0}^{\infty}\frac{\sin\sigma_k\zeta}{\sigma_k}\frac{\partial B_k}{\partial\xi} \qquad w^{(2)} = 0 \qquad (1.3)$$
$$\tau_{xz}^{(2)} = 2\mu\lambda\sum_{k=0}^{\infty}\cos\sigma_k\zeta\frac{\partial B_k}{\partial\eta}, \qquad \tau_{yz}^{(2)} = -2\mu\lambda\sum_{k=0}^{\infty}\cos\sigma_k\zeta\frac{\partial B_k}{\partial\xi}, \qquad \sigma_z^{(2)} = 0$$

$$\sigma_x^{(2)} = -\sigma_y^{(2)} = 4\mu\lambda^2 \sum_{k=0}^{\infty} \frac{\sin\sigma_k\zeta}{\sigma_k} \frac{\partial^2 B_k}{\partial\xi\partial\eta} \tag{1.4}$$

$$\tau_{yz}^{(2)} = 2\mu\lambda^2 \sum_{k=0}^{\infty} \frac{\sin\sigma_k\zeta}{\sigma_k} \left(\frac{\partial^2 B_k}{\partial\eta^2} - \frac{\partial^2 B_k}{\partial\xi^2}\right)$$

where $B_k(\xi, \eta)$ are found from the equations

$$\frac{\partial^2 B_k}{\partial\xi^2} + \frac{\partial^2 B_k}{\partial\eta^2} - \frac{\sigma_k^2}{\lambda^2} B_k = 0, \qquad \sigma_k = \frac{2k+1}{2}\pi \qquad (k = 0, 1, 2, \ldots) \tag{1.5}$$

The potential state of stress is given by the formulas

$$u^{(3)} = \lambda a \sum_{p=1}^{\infty} n_p(\zeta) \frac{\partial C_p}{\partial\xi}, \qquad v^{(3)} = \lambda a \sum_{p=1}^{\infty} n_p(\zeta) \frac{\partial C_p}{\partial\eta}, \qquad w^{(3)} = -a \sum_{p=1}^{\infty} q_p(\zeta) C_p \tag{1.6}$$

$$\tau_{xz}^{(3)} = 2\mu \sum_{p=1}^{\infty} r_p(\zeta) \frac{\partial C_p}{\partial\xi}, \qquad \tau_{yz}^{(3)} = 2\mu \sum_{p=1}^{\infty} r_p(\zeta) \frac{\partial C_p}{\partial\eta}$$

$$\sigma_z^{(3)} = \frac{2\mu}{\lambda} \sum_{p=1}^{\infty} t_p(\zeta) C_p, \qquad \tau_{xy}^{(3)} = 2\mu\lambda \sum_{p=1}^{\infty} n_p(\zeta) \frac{\partial^2 C_p}{\partial\xi\,\partial\eta} \tag{1.7}$$

$$\sigma_x^{(3)} = 2\mu \left[\frac{\nu-1}{\lambda} \sum_{p=1}^{\infty} s_p(\zeta) C_p + \lambda \sum_{p=1}^{\infty} n_p(\zeta) \frac{\partial^2 C_p}{\partial\xi^2}\right]$$

$$\sigma_y^{(3)} = 2\mu \left[\frac{\nu-1}{\lambda} \sum_{p=1}^{\infty} s_p(\zeta) C_p + \lambda \sum_{p=1}^{\infty} n_p(\zeta) \frac{\partial^2 C_p}{\partial\eta^2}\right]$$

where

$$n_p(\zeta) = \sin\gamma_p\zeta \left(\nu\sin\gamma_p + \frac{\cos\gamma_p}{\gamma_p}\right) + \nu\zeta\cos\gamma_p\cos\gamma_p\zeta$$

$$q_p(\zeta) = \cos\gamma_p\zeta\,[(1+\nu)\cos\gamma_p - \nu\gamma_p\sin\gamma_p] + \nu\gamma_p\zeta\cos\gamma_p\sin\gamma_p\zeta \tag{1.8}$$

$$s_p(\zeta) = \gamma_p\cos\gamma_p\sin\gamma_p\zeta, \qquad r_p(\zeta) = \nu\gamma_p(\sin\gamma_p\cos\gamma_p\zeta - \zeta\cos\gamma_p\sin\gamma_p\zeta)$$

$$t_p(\zeta) = \nu\gamma_p^2\left[\sin\gamma_p\zeta\left(\frac{\cos\gamma_p}{\gamma_p} - \sin\gamma_p\right) - \zeta\cos\gamma_p\cos\gamma_p\zeta\right]$$

In formulas (1.6) to (1.7), $C_p(\xi, \eta)$ are found from the equations

$$\frac{\partial^2 C_p}{\partial\xi^2} + \frac{\partial^2 C_p}{\partial\eta^2} - \frac{\gamma_p^2}{\lambda^2} C_p = 0 \qquad (p = 1, 2, \ldots) \tag{1.9}$$

where $2\gamma_p$ are the roots of the function $(\sin x)/x - 1$.

The summation in (1.6) to (1.7) extends over those roots γ_p that have positive real parts. Formulas (1.3) and (1.6) show that the displacements

u and v in the rotational case will be the components of the rotor [curl] of some function, and in the potential case they are those of the gradient of some function.

The formulated problem on the state of stress in the plate could be solved, if those boundary values of ψ, B_k and C_k necessary for a complete derivation are determined by the prescribed stresses on Γ_2. This will be done below with the aid of Lagrange's principle of virtual displacements. As a preliminary, we will discuss the degree of arbitrariness in the determination of ψ, B_k and C_k. Let us assume that all stresses in the plate are zero. By virtue of the independence of the state of stress, the biharmonic, the rotational and the potential states of stress must vanish separately. If the biharmonic state of stress vanishes, then, from (1.2), we have

$$2\nu\frac{\partial^2\psi}{\partial\xi^2}+(\nu-1)\frac{\partial^2\psi}{\partial\eta^2}=0\,,\qquad(\nu-1)\frac{\partial^2\psi}{\partial\xi^2}+2\nu\frac{\partial^2\psi}{\partial\eta^2}=0,\qquad\frac{\partial^2\psi}{\partial\xi\,\partial\eta}=0 \quad (1.10)$$

For values of ν that are physically possible, it follows from (1.10) that all second derivatives of ψ vanish, and consequently it has the form $\psi = k_1\xi + k_2\eta + k_3$. It is easy to see that this simply corresponds to a motion of the plate as a rigid body. In order to henceforth exclude this motion, we will assume that

$$w^{(1)}=0,\qquad\frac{\partial u^{(1)}}{\partial\zeta}=0,\qquad\frac{\partial v^{(1)}}{\partial\zeta}=0 \text{ when } \xi=\eta=\zeta=0 \quad (1.11)$$

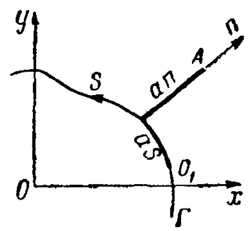

Fig. 2.

An analogous treatment shows that B_k and C_p vanish, if the corresponding stresses vanish. Thus, there is no arbitrariness at all in the determination of B_k and C_p.

2. For the following, certain formulas will be needed which characterize the behavior of the solutions of equations (1.5) and (1.9) for small values of h. Both of these equations can be written out in the following form:

$$\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}-\frac{\alpha^2}{\lambda^2}\Phi=0,\qquad \operatorname{Re}\alpha>0 \quad (2.1)$$

The Dirichlet problem for equation (2.1) will be treated. For the derivation of the relations that characterize this problem for small values of h, we pass over to the local system of dimensionless coordinates s and n (Fig. 2).

In this connection, equation (2.1) assumes the form

$$\frac{\partial^2\Phi}{\partial n^2}+\frac{\partial^2\Phi}{\partial s^2}\frac{R^2}{(R+an)^2}+\frac{\partial\Phi}{\partial n}\frac{a}{R+an}+\frac{\partial\Phi}{\partial s}\frac{anRR_s'}{(R+an)^3}-\frac{\alpha^2}{\lambda^2}\Phi=0 \qquad (2.2)$$

where R is the radius of curvature of contour Γ of the boundary Γ_1 (Fig. 1).

In order to obtain the asymptotic representation of the solution to (2.2) as $\lambda\to 0$, we will apply a method that was described in [7]. For this, we will carry out a stretching along the normals by setting $n=\lambda t$. In this connection, equation (2.2) assumes the form

$$\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial^2\Phi}{\partial s^2}\frac{\lambda^2R^2}{(R+a\lambda t)^2}+\frac{\partial\Phi}{\partial t}\frac{a\lambda}{R+a\lambda t}+\frac{\partial\Phi}{\partial s}\frac{at\lambda^3RR_s'}{(R+a\lambda t)^3}-\alpha^2\Phi=0 \qquad (2.3)$$

We will look for the solution of (2.3) in the form

$$\Phi(s,t)=\chi(s,t)\exp\alpha t \qquad (2.4)$$

If (2.4) is substituted into (2.3), for $\chi(s, t)$ we obtain

$$\frac{\partial^2\chi}{\partial t^2}+\frac{\partial^2\chi}{\partial s^2}\frac{\lambda^2R^2}{(R+a\lambda t)^2}+\frac{\partial\chi}{\partial t}\left(2\alpha+\frac{a\lambda}{R+a\lambda t}\right)+\frac{\partial\chi}{\partial s}\frac{at\lambda^3RR_s'}{(R+a\lambda t)^3}+\chi\frac{\alpha a\lambda}{R+a\lambda t}=0 \qquad (2.5)$$

Further, we will assume that $\chi(s, t)$ can be given by the series

$$\chi(s,t)=\chi_0(s,t)+\lambda\chi_1(s,t)+\lambda^2\chi_2(s,t)+\ldots \qquad (2.6)$$

From (2.5), we obtain for $\chi_k(s, t)$ the recurrence relations

$$\begin{gathered}\chi_{0tt}+2\alpha\chi_{0t}=0,\qquad \chi_{1tt}+2\alpha\chi_{1t}=-\frac{\alpha a\chi_0}{R}-\frac{1}{R}\chi_{0t}\\ \chi_{2tt}+2\alpha\chi_{2t}=-\chi_{0ss}-\frac{a}{R}\left(-\frac{at}{R}\chi_{0t}+\chi_{1t}-\frac{\alpha at}{R}\chi_{0t}+\alpha\chi_1\right)\\ \chi_{3tt}+2\alpha\chi_{3t}=-\chi_{0s}t\frac{aR_s'}{R^2}-\chi_{1ss}+2t\frac{a}{R}\chi_{0ss}-\frac{a}{R}\left(\chi_{2t}-t\frac{a}{R}\chi_{1t}+\right.\\ \left.+t^2\frac{a^2}{R^2}\chi_{0t}+\alpha\chi_2-\alpha t\frac{a}{R}\chi_1+\alpha t^2\frac{a^2}{R^2}\chi_0\right)\end{gathered} \qquad (2.7)$$

From (2.7), it is found that

$$\begin{gathered}\chi_0=\chi_0(s),\qquad \chi_1=-\frac{\chi_0at}{2R},\qquad \chi_2=\frac{1}{2\alpha}\left(\frac{3\alpha a^2}{4R^2}\chi_0t^2-\chi_0''t-\frac{a^2}{4R^2}\chi_0t\right)\\ \chi_3=\frac{1}{2\alpha}\left\{-\frac{5\alpha a^3}{8R^3}\chi_0t^3+t^2\left[\frac{3a}{2R}\chi_0''-\frac{aR'}{R^2}\chi_0'+\frac{a\chi_0}{4R^3}\left(2R'^2-RR''+\frac{3}{2}a^2\right)\right]+\right.\\ \left.+\frac{t}{2\alpha}\left[-\frac{2a}{R}\chi_0''+\frac{2aR'}{R^2}\chi_0'-\frac{a}{2R^3}\chi_0\left(2R'^2-RR''+a^2\right)\right]\right\}\end{gathered} \qquad (2.8)$$

where $\chi_0(s)$ is some, as yet arbitrary, sufficiently smooth function which is determined from the boundary conditions when $t = 0$. Let $\Phi(s, t)|_\Gamma=\varphi(s)$. In this case, (2.4), (2.6) and (2.8) yield $\chi_0(s)=\varphi(s)$.

Thus, for $\Phi(s, n)$ we obtain the asymptotic representation

$$\Phi(s, n) = \Big\{\varphi - \frac{a}{2R} n\varphi + \frac{1}{2\alpha}\Big[\frac{3a^2\alpha}{4R^2} n^2\varphi - \frac{a^2}{4R^2}\lambda n\varphi - \lambda n\varphi''\Big] + \\ + \frac{1}{2\alpha}\Big[-\frac{5a^3}{8R^3}\alpha n^3\varphi + \lambda n^2\Big(\frac{3a}{2R}\varphi'' - \frac{aR'}{R^2}\varphi' + a\frac{2R'^2 - RR'' + {}^3/_2 a^2}{4R^3}\varphi\Big) + \\ + \frac{\lambda^2 n}{2\alpha}\Big(-\frac{2a}{R}\varphi'' + \frac{2aR'}{R^2}\varphi' - a\frac{2R'^2 - RR'' + a^2}{2R^3}\varphi\Big)\Big] + \ldots\Big\}\exp\frac{\alpha n}{\lambda} \qquad (2.9)$$

Solution (2.9) has the property that $\Phi \to 0$ as the point moves into the region $(n \to -\infty)$. In the following, it will be necessary to have the expressions for the derivatives along the normal to the contour. From (2.9) it follows that

$$\frac{\partial\Phi}{\partial n}\Big|_\Gamma = \frac{1}{\lambda}\Big\{\alpha\varphi - \frac{a}{2R}\varphi\lambda - \Big[\frac{a^2}{2\alpha R^2}\varphi + \frac{\varphi''}{2\alpha}\Big]\lambda^2 + \\ + \Big[-\frac{a}{2\alpha^2 R}\varphi'' + \frac{aR'}{2\alpha^2 R^2}\varphi' - a\frac{2R'^2 - RR'' + a^2}{8\alpha^2 R^3}\varphi\Big]\lambda^3 + \ldots\Big\} \qquad (2.10)$$

$$\frac{\partial^2\Phi}{\partial n^2}\Big|_\Gamma = \frac{1}{\lambda^2}\Big\{\alpha^2\varphi - \frac{\alpha a}{R}\varphi\lambda + \Big[\frac{a^2}{2R^2}\varphi - \varphi''\Big]\lambda^2 + \Big[\frac{a}{2\alpha R}\varphi'' + \frac{a^3}{8\alpha R^3}\varphi\Big]\lambda^3 + \ldots\Big\}$$

We will assume that on the boundary $\varphi(s)$ can be represented in the form of the series

$$\varphi(s) = \varphi_0(s) + \lambda\varphi_1(s) + \lambda^2\varphi_2(s) + \ldots \qquad (2.11)$$

$$\frac{\partial\Phi}{\partial n}\Big|_\Gamma = \frac{1}{\lambda}\Big\{\alpha\varphi_0 + \lambda\Big(\alpha\varphi_1 - \frac{a}{2R}\varphi_0\Big) + \lambda^2\Big[\alpha\varphi_2 - \frac{a}{2R}\varphi_1 - \\ - \Big(\frac{a^2}{8\alpha R^2}\varphi_0 - \frac{\varphi_0''}{2\alpha}\Big)\Big] + \lambda^3\Big[\alpha\varphi_3 - \frac{a}{2R}\varphi_2 - \\ - \Big(\frac{a^2}{8\alpha R^2}\varphi_1 + \frac{\varphi_1''}{2\alpha}\Big) + \Big(-\frac{a}{2\alpha^2 R}\varphi_0'' + \frac{aR'}{2\alpha^2 R^2}\varphi_0' - a\frac{2R'^2 - RR'' + a^2}{8\alpha^2 R^3}\varphi_0\Big)\Big] + \ldots\Big\} \qquad (2.12)$$

$$\frac{\partial^2\Phi}{\partial n^2}\Big|_\Gamma = \frac{1}{\lambda^2}\Big\{\alpha^2\varphi_0 + \lambda\Big(\alpha^2\varphi_1 - \frac{\alpha a}{R}\varphi_0\Big) + \lambda^2\Big[\alpha^2\varphi_2 - \frac{\alpha a}{R}\varphi_1 + \\ + \Big(\frac{a^2}{2R^2}\varphi_0 - \varphi_0''\Big)\Big] + \\ + \lambda^3\Big[\alpha^2\varphi_3 - \frac{\alpha a}{R}\varphi_2 + \Big(\frac{a^2}{2R^2}\varphi_1 - \varphi_1''\Big) + \Big(\frac{a}{2\alpha R}\varphi_0'' + \frac{a^3}{8\alpha R^3}\varphi_0\Big)\Big] + \ldots\Big\} \qquad (2.13)$$

3. We will now make more precise assumptions concerning the external loading and the contour Γ. We will assume that the system of tractions $N(z, s)$, $T(z, s)$ and $Z(z, s)$, (Fig. 3) is prescribed at every point of Γ_2. We will write out the conditions that the moments of these forces about the x- and y-axes should vanish. We have

$$\oint_\Gamma \int_{-h}^{h} [-Tz\cos nx - Nz\sin nx + Zy]\, dz\, ds = 0$$

$$\oint_{\Gamma} \int_{-h}^{s} [-Tz \sin nx + Nz \cos nx - Zx]\, dz\, ds = 0 \qquad (3.1)$$

From (3.1), it is clear that the order of Z in $\lambda = h/a$ must be one unit higher than that of T and N.

We introduce the following statical characteristics of the external loading:

$$\int_{-h}^{h} Nz\, dz = h^2 M_1(s), \qquad \int_{-h}^{h} Nz^3\, dz = h^4 M_3(s), \qquad \int_{-h}^{h} Tz\, dz = h^2 G_1(s)$$

$$\int_{-h}^{h} Tz^3\, dz = h^4 G_3(s), \qquad \int_{-1}^{1} T \frac{\sin \sigma_m \zeta}{\sigma_m} d\zeta = T_m(s) \qquad (3.2)$$

$$\int_{-h}^{h} Z dz = h Q_0(s), \qquad \int_{-h}^{h} Zz^2\, dz = h^3 Q_2(s), \qquad \int_{-1}^{1} N \frac{\sin \sigma_m \zeta}{\sigma_m} d\zeta = N_m(s)$$

$$\int_{-1}^{1} N n_t(\zeta)\, d\zeta = N_t(s), \qquad \int_{-1}^{1} T n_t(\zeta)\, d\zeta = T_t(s), \qquad \int_{-1}^{1} Z q_t(\zeta)\, d\zeta = Z_t(s)$$

We will assume that all these statical characteristics can be represented in the form of power series in λ. In addition, it is evident that the expansion of M_1 will be of the form

$$M_1(s) = \lambda M_{11}(s) + \lambda^2 M_{12}(s) + \lambda^3 M_{13}(s) + \ldots \qquad (3.3)$$

It is obvious that M_3, G_1, G_3, N_m, T_m, N_t and T_t will have expansions of exactly the same type. The expansions of Q_0, Q_z and Z_t will commence with terms of the second order. We will further assume that all of these functions have a sufficient number of derivatives with respect to s.

We will also assume that the contour Γ bounds a simply-connected region Ω, is sufficiently smooth, and let the radius of curvature R of contour Γ as function of s have a sufficient number of derivatives.

4. We will determine the boundary conditions for ψ, B_k and C_p by making use of the fact that the stresses are known on the boundary of the plate. We will start from Lagrange's principle of virtual displacements, which in the present case can be written down in the form

$$\delta \iiint_V W dx\, dy\, dz - \iint_{\Gamma_2} (N \delta u_n + T \delta u_s + Z \delta w)\, d\sigma = 0 \qquad (4.1)$$

where u_n and u_s are the components of the displacements along the n- and s-axes.

We will seek the deformed state of the plate in the form

$$u = u^{(1)} + u^{(2)} + u^{(3)}$$
$$v = v^{(1)} + v^{(2)} + v^{(3)}$$
$$w = w^{(1)} + w^{(2)} + w^{(3)} \qquad (4.2)$$

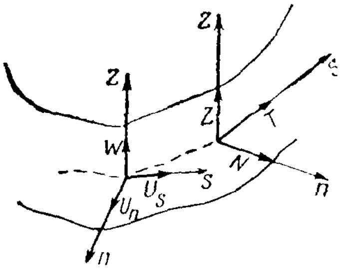

Fig. 3.

taking for the generalized displacements the values of the functions ψ, B_k and C_p and the value of the normal derivative of function ψ on Γ. We will substitute (4.2) into (4.1) and take account of the fact that the displacements (4.2) must be exact solutions of the equations of the theory of elasticity. As is well known [8], the volume integral on the left-hand side of (4.1) can in this case be converted into a surface integral

$$\iint_{\Gamma_2} (\sigma_n \, \delta u_n + \tau_{ns} \, \delta u_s + \tau_{nz} \, \delta w) \, d\sigma = \iint_{\Gamma_2'} (N \delta u_n + T \delta u_s + Z \delta w) \, d\sigma \quad (4.3)$$

where σ_n, τ_{ns}, τ_{nz}, u_n and u_s are defined by the formulas

$$\sigma_n = (\sigma_x^{(1)} + \sigma_x^{(2)} + \sigma_x^{(3)}) \, l^2 + (\sigma_y^{(1)} + \sigma_y^{(2)} + \sigma_y^{(3)}) \, m^2 + \\ + 2 \, (\tau_{xy}^{(1)} + \tau_{xy}^{(2)} + \tau_{xy}^{(3)}) \, lm$$

$$\tau_{ns} = (\tau_{xy}^{(1)} + \tau_{xy}^{(2)} + \tau_{xy}^{(3)}) \, (l^2 - m^2) + (\sigma_y^{(1)} + \sigma_y^{(2)} + \\ + \sigma_y^{(3)} - \sigma_x^{(1)} - \sigma_x^{(2)} - \sigma_x^{(3)}) \, lm \qquad (4.4)$$

$$\tau_{nz} = (\tau_{xz}^{(1)} + \tau_{xz}^{(2)} + \tau_{xz}^{(3)}) \, l + (\tau_{yz}^{(1)} + \tau_{yz}^{(2)} + \tau_{yz}^{(3)}) \, m$$

$$u_n = (u^{(1)} + u^{(2)} + u^{(3)}) \, l + (v^{(1)} + v^{(2)} + v^{(3)}) \, m$$
$$u_s = -(u^{(1)} + u^{(2)} + u^{(3)}) \, m + (v^{(1)} + v^{(2)} + v^{(3)}) \, l \qquad (4.5)$$

All stresses on the right-hand side of (4.4) should be taken from (1.2), (1.4) and (1.7), and in (4.5) the displacements should be taken from (1.1), (1.3) and (1.6). In this connection, we obtain

$$\sigma_n = 2\mu\lambda \left\{ \left[2\nu \frac{\partial^2 \psi}{\partial n^2} + (\nu - 1) \left(\frac{1}{H^2} \frac{\partial^2 \psi}{\partial s^2} + \frac{a}{R} \frac{1}{H} \frac{\partial \psi}{\partial n} + n \frac{aR'}{R^2} \frac{1}{H^3} \frac{\partial \psi}{\partial s} \right) \right] \zeta - \right.$$

$$\left. - \left(\nu + \frac{1}{3} \right) \frac{\zeta^3}{2} \lambda^2 \frac{\partial^2 \Delta \psi}{\partial n^2} \right\} + 4\mu\lambda^2 \sum_{k=0}^{\infty} \frac{\sin \sigma_k \zeta}{\sigma_k} \left(\frac{1}{H} \frac{\partial^2 B_k}{\partial n \, \partial s} - \frac{1}{H^2} \frac{a}{R} \frac{\partial B_k}{\partial s} \right) +$$

$$+ 2\mu \left[\frac{\nu - 1}{\lambda} \sum_{p=1}^{\infty} s_p(\zeta) C_p + \lambda \sum_{p=1}^{\infty} n_p(\zeta) \frac{\partial^2 C_n}{\partial n^2} \right]$$

$$\tau_{ns} = 2\mu\lambda \left\{ (\nu + 1) \zeta \left(\frac{1}{H} \frac{\partial^2 \psi}{\partial n \, \partial s} - \frac{1}{H^2} \frac{a}{R} \frac{\partial \psi}{\partial s} \right) - \left(\nu + \frac{1}{3} \right) \frac{\zeta^3}{2} \lambda^2 \left(\frac{1}{H} \frac{\partial^2 \Delta\psi}{\partial n \partial s} - \right.\right.$$

$$\left.\left. - \frac{1}{H^2} \frac{a}{R} \frac{\partial \Delta\psi}{\partial s} \right) \right\} + 2\mu\lambda^2 \sum_{k=0}^{\infty} \frac{\sin \sigma_k \zeta}{\sigma_k} \left(\frac{1}{H^2} \frac{\partial^2 B_k}{\partial s^2} + \frac{1}{H} \frac{a}{R} \frac{\partial B_k}{\partial n} + \right.$$

$$\left. + n \frac{aR'}{R^2} \frac{1}{H^3} \frac{\partial B_k}{\partial s} - \frac{\partial^2 B_k}{\partial n^2} \right) + 2\mu\lambda \sum_{p=1}^{\infty} n_p(\zeta) \left(\frac{1}{H} \frac{\partial^2 C_p}{\partial n \, \partial s} - \frac{1}{H^2} \frac{a}{R} \frac{\partial C_p}{\partial s} \right) \quad (4.6)$$

$$\tau_{nz} = 2\mu\nu\lambda^2 (1 - \zeta^2) \frac{\partial \Delta\psi}{\partial n} + 2\mu\lambda \sum_{k=0}^{\infty} \cos \sigma_k \zeta \frac{\partial B_k}{\partial s} + 2\mu \sum_{p=1}^{\infty} r_p(\zeta) \frac{\partial C_p}{\partial n}$$

$$u_n = (\nu + 1) a\lambda\zeta \frac{\partial \psi}{\partial n} - \left(\nu + \frac{1}{3} \right) \frac{\zeta^3}{2} \lambda^3 a \frac{\partial \Delta\psi}{\partial n} +$$

$$+ 2\lambda^2 a \sum_{k=0}^{\infty} \frac{\sin \sigma_k \zeta}{\sigma_k} \frac{1}{H} \frac{\partial B_k}{\partial s} + \lambda a \sum_{p=1}^{\infty} n_p(\zeta) \frac{\partial C_p}{\partial n}$$

$$u_s = (\nu + 1) a\lambda\zeta \frac{1}{H} \frac{\partial \psi}{\partial s} - \left(\nu + \frac{1}{3} \right) \frac{\zeta^3}{2} \lambda^3 a \frac{1}{H} \frac{\partial \Delta\psi}{\partial s} - \qquad \left(H = 1 + n\frac{a}{R} \right)$$

$$- 2\lambda^2 a \sum_{k=0}^{\infty} \frac{\sin \sigma_k \zeta}{\sigma_k} \frac{\partial B_k}{\partial n} + \lambda a \sum_{p=1}^{\infty} n_p(\zeta) \frac{1}{H} \frac{\partial C_p}{\partial s} \quad (4.7)$$

$$w = -(\nu + 1) a\psi - (\nu - 1) \frac{\zeta^2}{2} \lambda^2 a \Delta\psi + 2\nu\lambda^2 a \Delta\psi - a \sum_{p=1}^{\infty} q_p(\zeta) C_p$$

The values of the functions $B_k(s, n)$ and $C_p(s, n)$ on Γ will be denoted by $b_k(s)$ and $c_p(s)$, respectively. In equation (4.3), we will only vary the boundary value of ψ, by taking $\delta b_k = \delta c_p = 0$. In this case we obtain

$$2\mu \oint_{\Gamma} ds \left\{ A\delta \frac{\partial \psi}{\partial n} + B\delta \frac{\partial \Delta\psi}{\partial n} + C\delta \frac{\partial \psi}{\partial s} + D\delta \frac{\partial \Delta\psi}{\partial s} + E\delta\psi + F\delta\Delta\psi \right\} =$$

$$= \oint_{\Gamma} ds \left\{ (\nu + 1) \lambda M_1 \delta \frac{\partial \psi}{\partial n} - \frac{1}{2} \left(\nu + \frac{1}{3} \right) \lambda^3 M_3 \delta \frac{\partial \Delta\psi}{\partial n} + (\nu + 1) \lambda G_1 \delta \frac{\partial \psi}{\partial s} - \right.$$

$$\left. - \frac{1}{2} \left(\nu + \frac{1}{3} \right) \lambda^3 G_3 \delta \frac{\partial \Delta\psi}{\partial s} - (\nu + 1) Q_0 \delta\psi + \lambda^2 \left(\frac{\nu - 1}{2} Q_2 + 2\nu Q_0 \right) \delta\Delta\psi \right\} \quad (4.8)$$

where

$$A = \left\{\frac{2}{3}(\nu + 1)\lambda^2\left[2\nu\frac{\partial^2\psi}{\partial n^2} + (\nu - 1)\left(\frac{\partial^2\psi}{\partial s^2} + \frac{a}{R}\frac{\partial\psi}{\partial n}\right)\right] - \right.$$

$$- \frac{1}{5}\left(\nu + \frac{1}{3}\right)(\nu + 1)\lambda^4\frac{\partial^2\Delta\psi}{\partial n^2} + 4(\nu + 1)\lambda^3\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^3}\left(\frac{\partial^2 B_k}{\partial n\partial s} - \frac{a}{R}\frac{\partial b_k}{\partial s}\right) +$$

$$\left. + 2(\nu^2 - 1)\sum_{p=1}^{\infty}\sin^2\gamma_p\left(c_p - \frac{\lambda^2}{\gamma_p^2}\frac{\partial^2 C_p}{\partial n^2}\right)\right\}_{n=0} \qquad (4.9)$$

$$B = \left\{-\frac{1}{5}\left(\nu + \frac{1}{3}\right)\lambda^4\left[2\nu\frac{\partial^2\psi}{\partial n^2} + (\nu - 1)\left(\frac{\partial^2\psi}{\partial s^2} + \frac{a}{R}\frac{\partial\psi}{\partial n}\right)\right] + \right.$$

$$+ \frac{1}{14}\left(\nu + \frac{1}{3}\right)^2\lambda^6\frac{\partial^2\Delta\psi}{\partial n^2} - 2\left(\nu + \frac{1}{3}\right)\lambda^5\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(3\sigma_k^2 - 6)}{\sigma_k^5}\left(\frac{\partial^2 B_k}{\partial n\partial s} - \frac{a}{R}\frac{\partial b_k}{\partial s}\right) +$$

$$+ (1 - \nu)\left(\nu + \frac{1}{3}\right)\lambda^2\sum_{p=1}^{\infty}\left(\frac{\gamma_p^2 - 6}{\gamma_p^2}\sin^2\gamma_p + 2\right)\left(c_p - \frac{\lambda^2}{\gamma_p^2}\frac{\partial^2 C_p}{\partial n^2}\right) -$$

$$\left. - 12\nu\left(\nu + \frac{1}{3}\right)\lambda^4\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p^4}\frac{\partial^2 C_p}{\partial n^2}\right\}_{n=0} \qquad (4.10)$$

$$C = \left\{\frac{2}{3}(\nu + 1)^2\lambda^2\left(\frac{\partial^2\psi}{\partial n\partial s} - \frac{a}{R}\frac{\partial\psi}{\partial s}\right) - \frac{1}{5}(\nu + 1)\left(\nu + \frac{1}{3}\right)\lambda^4\left(\frac{\partial^2\Delta\psi}{\partial n\partial s} - \frac{a}{R}\frac{\partial\Delta\psi}{\partial s}\right) + \right.$$

$$+ 2(\nu + 1)\lambda^3\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^3}\left(\frac{\partial^2 b_k}{\partial s^2} + \frac{a}{R}\frac{\partial B_k}{\partial n} - \frac{\partial^2 B_k}{\partial n^2}\right) -$$

$$\left. - 2(\nu^2 - 1)\lambda^2\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p^2}\left(\frac{\partial^2 C_p}{\partial s\partial n} - \frac{a}{R}\frac{\partial c_p}{\partial s}\right)\right\}_{n=0} \qquad (4.11)$$

$$D = \left\{-\frac{1}{5}(\nu + 1)\left(\nu + \frac{1}{3}\right)\lambda^4\left(\frac{\partial^2\psi}{\partial n\partial s} - \frac{a}{R}\frac{\partial\psi}{\partial s}\right) + \right.$$

$$+ \frac{1}{14}\left(\nu + \frac{1}{3}\right)^2\lambda^6\left(\frac{\partial^2\Delta\psi}{\partial n\partial s} - \frac{a}{R}\frac{\partial\Delta\psi}{\partial s}\right) - \left(\nu + \frac{1}{3}\right)\lambda^5\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{\partial^2 b_k}{\partial s^2} + \right.$$

$$\left. + \frac{a}{R}\frac{\partial B_k}{\partial n} - \frac{\partial^2 B_k}{\partial n^2}\right)\frac{3\sigma_k^2 - 6}{\sigma_k^5} - \left(\nu + \frac{1}{3}\right)\lambda^4\sum_{p=1}^{\infty}\frac{1}{\gamma_p^2}\left[\left(\frac{\gamma_p^2 - 6}{\gamma_p^2}\sin^2\gamma_p + 2\right)(1 - \nu) + \right.$$

$$\left.\left. + 12\nu\frac{\sin^2\gamma_p}{\gamma_p^2}\right]\left(\frac{\partial^2 C_p}{\partial n\partial s} - \frac{a}{R}\frac{\partial c_p}{\partial s}\right)\right\}_{n=0} \qquad (4.12)$$

$$E = -\frac{4}{3}\nu(\nu + 1)\lambda^2\frac{\partial\Delta\psi}{\partial n}\bigg|_{n=0} - 2(\nu + 1)\lambda\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\sigma_k}\frac{\partial b_k}{\partial s} \qquad (4.13)$$

$$F=\left\{\frac{2}{15}\nu\,(19\nu+1)\,\lambda^4\frac{\partial\Delta\psi}{\partial n}+\lambda^3\sum_{k=0}^{\infty}\left[4\nu-(\nu-1)\frac{\sigma_k^2-2}{\sigma_k^2}\right]\frac{(-1)^{k+1}}{\sigma_k}\frac{\partial b_k}{\partial s}-\right.$$
$$\left.-\,4\nu\,(\nu-1)\,\lambda^2\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p^2}\frac{\partial C_p}{\partial n}\right\}_{n=0} \qquad (4.14)$$

If some of the terms in relation (4.8) are integrated by parts, it can be given the following form:

$$2\mu\oint_\Gamma ds\left\{A\delta\frac{\partial\psi}{\partial n}+B\delta\frac{\partial\Delta\psi}{\partial n}+\left(E-\frac{\partial C}{\partial s}\right)\delta\psi+\left(F-\frac{\partial D}{\partial s}\right)\delta\Delta\psi\right\}=$$
$$=\oint_\Gamma ds\left\{(\nu+1)\,\lambda M_1\delta\frac{\partial\psi}{\partial n}-\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3M_3\delta\frac{\partial\Delta\psi}{\partial n}-(\nu+1)\left(Q_0+\lambda\frac{\partial G_1}{\partial s}\right)\delta\psi+\right.$$
$$\left.+\left[\lambda^2\left(\frac{\nu-1}{2}Q_2+2\nu Q_0\right)+\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\frac{\partial G_3}{\partial s}\right]\delta\Delta\psi\right\} \qquad (4.15)$$

We assume that $\psi|_\Gamma$ and $\partial\psi/\partial n|_\Gamma$ are independent in the variational equation (4.15). Since ψ is a harmonic function, $\Delta\psi$ and $\partial\Delta\psi/\partial n$ on Γ can be expressed in terms of $\psi|_\Gamma$ and $\partial\psi/\partial n|_\Gamma$. In concrete terms, this expression can be established in the following way: it is known that every biharmonic function can be determined from its values and the values of the normal derivative on the boundary curve by means of the formula

$$\psi\,(P)=\oint_\Gamma K_1\,(P,\;Q)\,\psi\,(Q)\,dQ+\oint_\Gamma K_2\,(P,\;Q)\,\frac{\partial\psi}{\partial n}(Q)\,dQ \qquad (4.16)$$

where K_1 and K_2 can be expressed in terms of Green's function $G(P,\ Q)$ of the first boundary-value problem for the biharmonic equation. From (4.16) it follows that

$$\Delta\psi\,|_\Gamma=\oint_\Gamma K_{11}\,(s,\sigma)\,\psi\,(\sigma,\,0)\,d\sigma+\oint_\Gamma K_{12}\,(s,\sigma)\,\frac{\partial\psi\,(\sigma,\,n)}{\partial n}\Big|_{n=0}d\sigma \qquad (4.17)$$

$$\frac{\partial}{\partial n}\,\Delta\psi\Big|_\Gamma=\oint_\Gamma K_{21}\,(s,\sigma)\,\psi\,(\sigma,\,0)\,d\sigma+\oint_\Gamma K_{22}\,(s,\sigma)\,\frac{\partial\psi\,(\sigma,\,n)}{\partial n}\Big|_{n=0}d\sigma \qquad (4.18)$$

Moreover, the functions K_{ij} can be expressed in terms of the derivatives of $G(P,\ Q)$, and they contain the delta-function of s, σ and its derivatives. However, for our reasoning in the following this will not be of importance. From (4.17) and (4.18), it is possible to express the variations $\delta\Delta\psi|_\Gamma$ and $\delta[\partial\Delta\psi/\partial n]_\Gamma$ in terms of $\delta\psi|_\Gamma$ and $\delta[\partial\psi/\partial n]_\Gamma$.

In addition, formula (4.15) assumes the form

$$2\mu\oint_\Gamma\left\{A(s)\,\delta\frac{\partial\psi(s,n)}{\partial n}\Big|_{n=0}+\left[E(s)-\frac{\partial C(s)}{\partial s}\right]\delta\psi(s,0)\right\}ds+$$
$$+2\mu\oint_\Gamma\oint_\Gamma\left\{B(s)K_{21}(s,\sigma)+\left[F(s)-\frac{\partial D(s)}{\partial s}\right]K_{11}(s,\sigma)\right\}\delta\psi(\sigma,0)\,ds\,d\sigma+$$
$$+2\mu\oint_\Gamma\oint_\Gamma\left\{B(s)K_{22}(s,\sigma)+\left[F(s)-\frac{\partial D(s)}{\partial s}\right]K_{12}(s,\sigma)\right\}\delta\frac{\partial\psi(\sigma,n)}{\partial n}\Big|_{n=0}ds\,d\sigma=$$
$$=\oint_\Gamma\left\{(\nu+1)\lambda M_1(s)\,\delta\frac{\partial\psi(s,n)}{\partial n}\Big|_{n=0}-(\nu+1)\left[Q_0(s)+\lambda\frac{\partial G_1(s)}{\partial s}\right]\delta\psi(s,0)\right\}ds+$$
$$+\oint_\Gamma\oint_\Gamma\left\{-\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3M_3(s)K_{21}(s,\sigma)+\left[\frac{\nu-1}{2}\lambda^2Q_2(s)+2\nu\lambda^2Q_0(s)+\right.\right.$$
$$\left.\left.\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\frac{\partial G_3(s)}{\partial s}\right]K_{11}(s,\sigma)\right\}\delta\psi(\sigma,0)\,ds\,d\sigma+$$
$$+\oint_\Gamma\oint_\Gamma\left\{-\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3M_3(s)K_{22}(s,\sigma)+\left[\frac{\nu-1}{2}\lambda^2Q_2(s)+2\nu\lambda^2Q_0(s)+\right.\right.$$
$$\left.\left.+\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\frac{\partial G_3(s)}{\partial s}\right]K_{12}(s,\sigma)\right\}\delta\frac{\partial\psi(\sigma,n)}{\partial n}\Big|_{n=0}ds\,d\sigma \tag{4.19}$$

If now the order of integration in the double integrals is reversed, and the coefficients of both $\delta[\partial\psi(s,n)/\partial n]_{n=0}$ and of $\delta\psi(s,0)$ on the right- and left-hand sides of (4.19) are equated, we obtain the following two functional equations for the determination of the boundary values of ψ, $\partial\psi/\partial n$, B_k and C_p:

$$A(s)+\oint_\Gamma B(\sigma)K_{22}(\sigma,s)\,d\sigma+\oint_\Gamma\left[F(\sigma)-\frac{\partial D(\sigma)}{\partial\sigma}\right]K_{12}(\sigma,s)\,d\sigma= \tag{4.20}$$
$$=\frac{1}{2\mu}(\nu+1)\lambda M_1(s)-\frac{1}{4\mu}\left(\nu+\frac{1}{3}\right)\lambda^3\oint_\Gamma M_3(\sigma)K_{22}(\sigma,s)\,d\sigma+$$
$$+\frac{1}{2\mu}\oint_\Gamma K_{12}(\sigma,s)\left\{\lambda^2\left[\frac{\nu-1}{2}Q_2(\sigma)+2\nu Q_0(\sigma)\right]+\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\frac{\partial G_3(\sigma)}{\partial\sigma}\right\}d\sigma$$

$$E(s)-\frac{\partial C(s)}{\partial s}+\oint_\Gamma B(\sigma)K_{21}(\sigma,s)\,d\sigma+\oint_\Gamma\left[F(\sigma)-\frac{\partial D(\sigma)}{\partial\sigma}\right]K_{11}(\sigma,s)\,d\sigma=$$
$$=-\frac{\nu+1}{2\mu}\left[Q_0(s)+\lambda\frac{\partial G_1(s)}{\partial s}\right]-\frac{1}{4\mu}\left(\nu+\frac{1}{3}\right)\lambda^3\oint_\Gamma M_3(\sigma)K_{21}(\sigma,s)\,d\sigma+$$
$$+\frac{1}{2\mu}\oint_\Gamma K_{11}(\sigma,s)\left\{\lambda^2\left[\frac{\nu-1}{2}Q_2(\sigma)+2\nu Q_0(\sigma)\right]+\frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\frac{\partial G_3(\sigma)}{\partial\sigma}\right\}d\sigma \tag{4.21}$$

5. We will now assume that the variation of all quantities in equation (4.3) can be achieved only at the expense of a variation of the boundary value B_m. We note that the term $\delta[\partial B_m/\partial n]_\Gamma$ occurring in the variational equation is expressed in terms of δb_m by means of relation (2.10). Finally, we have

$$\frac{2\lambda^2}{\sigma_m^{\,3}}(-1)^{m+1}\Big\{2\nu\frac{\partial^3\psi}{\partial s\,\partial n^2}\lambda+(\nu-1)\Big(\frac{\partial^3\psi}{\partial s^3}-\frac{aR'}{R^2}\frac{\partial\psi}{\partial n}+\frac{a}{R}\frac{\partial^2\psi}{\partial s\,\partial n}\Big)\lambda-$$
$$-\frac{1}{2}\Big(\nu+\frac{1}{3}\Big)\frac{3\sigma_m^{\,2}-6}{\sigma_m^{\,2}}\lambda^3\frac{\partial^3\Delta\psi}{\partial s\,\partial n^2}\Big\}_{n=0}+ \qquad (5.1)$$
$$+\frac{2\lambda^2}{\sigma_m^{\,3}}(-1)^{m+1}\Big\{\Big[\sigma_m-\frac{a\lambda}{2R}-\frac{\lambda^2}{2\sigma_m}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]\Big[(\nu+1)\Big(\frac{\partial^2\psi}{\partial s\partial n}-\frac{a}{R}\frac{\partial\psi}{\partial s}\Big)-$$
$$-\frac{1}{2}\Big(\nu+\frac{1}{3}\Big)\frac{3\sigma_m^{\,2}-6}{\sigma_m^{\,2}}\lambda^2\Big(\frac{\partial^2\Delta\psi}{\partial s\partial n}-\frac{a}{R}\frac{\partial\Delta\psi}{\partial s}\Big)\Big]\Big\}_{n=0}+$$
$$+\frac{\lambda^3}{\sigma_m^{\,2}}\Big\{2\Big(\frac{\partial^3B_m}{\partial s^2\partial n}+\frac{aR'}{R^2}\frac{\partial b_m}{\partial s}-\frac{a}{R}\frac{\partial^2b_m}{\partial s^2}\Big)\lambda+$$
$$+\Big[\sigma_m-\frac{a\lambda}{2R}-\frac{\lambda^2}{2\sigma_m}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]\Big[\frac{\partial^2b_m}{\partial s^2}-\frac{\partial^2B_m}{\partial n^2}+\frac{a}{R}\frac{\partial B_m}{\partial n}\Big]\Big\}_{n=0}-$$
$$-\frac{2\lambda^2(-1)^{m+1}}{\sigma_m}\sum_{p=1}^{\infty}\frac{\cos^2\gamma_p}{\gamma_p^{\,2}-\sigma_m^{\,2}}\Big\{\frac{\nu-1}{\lambda}\gamma_p^{\,2}\frac{\partial c_p}{\partial s}+\Big(1-\nu\frac{\gamma_p^{\,2}+\sigma_m^{\,2}}{\gamma_p^{\,2}-\sigma_m^{\,2}}\Big)\frac{\partial^3C_p}{\partial s\partial n^2}\lambda+$$
$$+\Big(1-\nu\frac{\gamma_p^{\,2}+\sigma_m^{\,2}}{\gamma_p^{\,2}-\sigma_m^{\,2}}\Big)\Big[\sigma_m-\frac{a\lambda}{2R}-\frac{\lambda^2}{2\sigma_m}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]\Big(\frac{\partial^2C_p}{\partial n\partial s}-$$
$$-\frac{a}{R}\frac{\partial c_p}{\partial s}\Big)\Big\}_{n=0}=\frac{\lambda}{2\mu}\Big\{\lambda\frac{\partial N_m}{\partial s}+\Big[\sigma_m-\frac{a\lambda}{2R}-\frac{\lambda^2}{2\sigma_m}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]T_m\Big\}$$
$$(m=0,1,2,\ldots).$$

If it is similarly agreed that δc_i in equation (4.3) is non-zero, we obtain the following functional equation:

$$\Big\{2(1-\nu^2)\lambda^4\frac{\sin^2\gamma_t}{\gamma_t^{\,2}}\Big(\frac{\partial^3\psi}{\partial s^2\partial n}+\frac{aR'}{R^2}\frac{\partial\psi}{\partial s}-\frac{a}{R}\frac{\partial^2\psi}{\partial s^2}\Big)-$$
$$-\Big(\nu+\frac{1}{3}\Big)\frac{\lambda^6}{\gamma_t^{\,2}}\Big[\Big(\frac{\gamma_t^{\,2}-6}{\gamma_t^{\,2}}\sin^2\gamma_t+2\Big)(1-\nu)+12\nu\frac{\sin^2\gamma_t}{\gamma_t^{\,2}}\Big]\times$$
$$\times\Big(\frac{\partial^3\Delta\psi}{\partial s^2\partial n}+\frac{aR'}{R^2}\frac{\partial\Delta\psi}{\partial s}-\frac{a}{R}\frac{\partial^2\Delta\psi}{\partial s^2}\Big)+12\nu\Big(\nu+\frac{1}{3}\Big)\lambda^4\frac{\sin^2\gamma_t}{\gamma_t^{\,2}}\frac{\partial\Delta\psi}{\partial n}\Big\}_{n=0}-$$
$$-\Big[\gamma_t-\frac{a\lambda}{2R}-\frac{\lambda^2}{2\gamma_t}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]\Big\{\Big[2\nu\frac{\partial^2\psi}{\partial n^2}+(\nu-1)\Big(\frac{\partial^2\psi}{\partial s^2}+\frac{a}{R}\frac{\partial\psi}{\partial n}\Big)\Big]\times$$
$$\times 2(1-\nu)\lambda^3\frac{\sin^2\gamma_t}{\gamma_t^{\,2}}-\Big(\nu+\frac{1}{3}\Big)\frac{\lambda^5}{\gamma_t^{\,2}}\Big[\Big(\frac{\gamma_t^{\,2}-6}{\gamma_t^{\,2}}\sin^2\gamma_t+2\Big)(1-\nu)+$$

$$
+12\nu \frac{\sin^2\gamma_t}{\gamma_t^2}\Big]\frac{\partial^2\Delta\psi}{\partial n^2}\Big\}_{n=0} - 2\lambda^2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\cos^2\gamma_t}{\sigma_k(\gamma_t^2-\sigma_k^2)}\Big\{\Big(1-\nu\frac{\gamma_t^2+\sigma_k^2}{\gamma_t^2-\sigma_k^2}\Big)\lambda^3 \times
$$

$$
\times\Big(\frac{\partial^3 b_k}{\partial s^3} - \frac{\partial^3 B_k}{\partial s\partial n^2} - \frac{aR'}{R^2}\frac{\partial B_k}{\partial n} + \frac{a}{R}\frac{\partial^2 B_k}{\partial s\partial n}\Big) +
$$

$$
+\sigma_k^2\Big(1-\nu\frac{\sigma_k^2-3\gamma_t^2}{\gamma_t^2-\sigma_k^2}\Big)\lambda\frac{\partial b_k}{\partial s} - 2\lambda^2\Big(1-\nu\frac{\gamma_t^2+\sigma_k^2}{\gamma_t^2-\sigma_k^2}\Big)\times
$$

$$
\times\Big[\gamma_t - \frac{a\lambda}{2R} - \frac{\lambda^2}{2\gamma_t}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]\Big(\frac{\partial^2 B_k}{\partial n\partial s} - \frac{a}{R}\frac{\partial b_k}{\partial s}\Big)\Big\}_{n=0} +
$$

$$
+2\lambda\sum_{p=1(p\neq t)}^{\infty}\frac{\cos^2\gamma_t - \cos^2\gamma_p}{\gamma_p^2-\gamma_t^2}\Big\{\lambda^3\Big[1-\nu^2-8\nu^2\frac{\gamma_p^2\gamma_t^2}{(\gamma_t^2-\gamma_p^2)^2}\Big]\times
$$

$$
\times\Big(\frac{\partial^3 C_p}{\partial s^2\partial n} + \frac{aR'}{R^2}\frac{\partial c_p}{\partial s} - \frac{a}{R}\frac{\partial^2 c_p}{\partial s^2}\Big) - 2\nu\lambda\frac{\gamma_p^2\gamma_t^2}{\gamma_t^2-\gamma_p^2}\Big(1+\nu\frac{\gamma_p^2+3\gamma_t^2}{\gamma_t^2-\gamma_p^2}\Big)\frac{\partial C_p}{\partial n} -
$$

$$
-\Big[\gamma_t - \frac{a\lambda}{2R} - \frac{\lambda^2}{2\gamma_t}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]\Big[(\nu-1)\gamma_p^2\Big(1-\nu\frac{\gamma_t^2+\gamma_p^2}{\gamma_t^2-\gamma_p^2}\Big)c_p +
$$

$$
+\lambda^2\Big(1-\nu^2-8\nu^2\frac{\gamma_p^2\gamma_t^2}{(\gamma_t^2-\gamma_p^2)^2}\Big)\frac{\partial^2 C_p}{\partial n^2}\Big\}_{n=0} +
$$

$$
+\nu\lambda\Big\{\lambda^3\Big(\nu+2-\frac{2}{3}\nu\cos^2\gamma_t\Big)\Big(\frac{\partial^3 C_t}{\partial s^2\partial n} + \frac{aR'}{R^2}\frac{\partial c_t}{\partial s} - \frac{a}{R}\frac{\partial^2 c_t}{\partial s^2}\Big) +
$$

$$
+\lambda\gamma_t^2\Big(1+\frac{2}{3}\nu\cos^2\gamma_t\Big)\frac{\partial C_t}{\partial n} - \Big[\gamma_t - \frac{a\lambda}{2R} - \frac{\lambda^2}{2\gamma_t}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]\times
$$

$$
\times\Big[(\nu-1)\gamma_t^2 c_t + \lambda^2\Big(\nu+2-\frac{2}{3}\nu\cos^2\gamma_t\Big)\frac{\partial^2 C_t}{\partial n^2}\Big]\Big\}_{n=0} =
$$

$$
= \frac{\lambda^2}{2\mu}\Big\{\lambda\frac{\partial T_t}{\partial s} + Z_t - \Big[\gamma_t - \frac{a\lambda}{2R} - \frac{\lambda^2}{2\gamma_t}\Big(\frac{a^2}{4R^2}+\frac{\partial^2}{\partial s^2}\Big)+\ldots\Big]N_t\Big\} \quad (t=1,2,3,\ldots) \tag{5.2}
$$

Thus, by taking $m = 0, 1, 2, \ldots;\ t = 1, 2, 3, \ldots$ in equations (5.1) to (5.2), we obtain in addition two denumerable systems of functional equations, which, instead of (4.20) and (4.21), form a system of $2\infty + 2$ equations with exactly the same number of unknowns, namely $\psi|_\Gamma$, $\partial\psi/\partial n|_\Gamma$; b_k and c_p.

6. We will now return to the problem of determining the boundary values of ψ, $\partial\psi/\partial n$, B_k and C_p. We will seek ψ, b_k and c_p in the form of the following series:

$$
\begin{aligned}
\psi(s,n) &= \psi_0(s,n) + \lambda\psi_1(s,n) + \lambda^2\psi_2(s,n) + \ldots \\
b_k(s) &= b_{k0}(s) + \lambda b_{k1}(s) + \lambda^2 b_{k2}(s) + \ldots \\
c_p(s) &= c_{p0}(s) + \lambda c_{p1}(s) + \lambda^2 c_{p2}(s) + \ldots
\end{aligned} \tag{6.1}
$$

Then, according to formulas (2.13) to (2.14) we have

$$\frac{\partial C_p}{\partial n}\bigg|_\Gamma = \frac{1}{\lambda}\Big\{\gamma_p c_{p0} + \lambda\Big(\gamma_p c_{p1} - \frac{a}{2R}c_{p0}\Big) + \lambda^2\Big[\gamma_p c_{p2} - \frac{a}{2R}c_{p1} - \Big(\frac{a^2}{8\gamma_p R^2}c_{p0} + \frac{c_{p0}''}{2\gamma_p}\Big)\Big] + \lambda^3\Big[\gamma_p c_{p3} - \frac{a}{2R}c_{p2} - \Big(\frac{a^2}{8\gamma_p R^2}c_{p1} + \frac{c_{p1}''}{2\gamma_p}\Big) + \Big(-\frac{a}{2\gamma_p^2 R}c_{p0}'' + \frac{aR'}{2\gamma_p^2 R^2}c_{p0}' - a\frac{2R'^2 - RR'' + a^2}{8\gamma_p^2 R^3}c_{p0}\Big)\Big] + \ldots\Big\} \quad (6.2)$$

$$\frac{\partial^2 C_p}{\partial n^2}\bigg|_\Gamma = \frac{1}{\lambda^2}\Big\{\gamma_p^2 c_{p0} + \lambda\Big(\gamma_p^2 c_{p1} - \frac{\gamma_p a}{R}c_{p0}\Big) + \lambda^2\Big[\gamma_p^2 c_{p2} - \frac{\gamma_p a}{R}c_{p1} + \Big(\frac{a^2}{2R^2}c_{p0} - c_{p0}''\Big)\Big] + \lambda^3\Big[\gamma_p^2 c_{p3} - \frac{\gamma_p a}{R}c_{p2} + \Big(\frac{a^2}{2R^2}c_{p1} - c_{p1}''\Big) + \Big(\frac{a}{2\gamma_p R}c_{p0}'' + \frac{a^3}{8\gamma_p R^3}c_{p0}\Big)\Big] + \ldots\Big\}$$

$$\frac{\partial B_k}{\partial n}\bigg|_\Gamma = \frac{1}{\lambda}\Big\{\sigma_k b_{k0} + \lambda\Big(\sigma_k b_{k1} - \frac{a}{2R}b_{k0}\Big) + \lambda^2\Big[\sigma_k b_{k2} - \frac{a}{2R}b_{k1} - \Big(\frac{a^2}{8\sigma_k R^2}b_{k0} + \frac{b_{k0}''}{2\sigma_k}\Big)\Big] + \lambda^3\Big[\sigma_k b_{k3} - \frac{a}{2R}b_{k2} - \Big(\frac{a^2}{8\sigma_k R^2}b_{k1} + \frac{b_{k1}''}{2\sigma_k}\Big) + \Big(-\frac{a}{2\sigma_k^2 R}b_{k0}'' + \frac{aR'}{2\sigma_k^2 R^2}b_{k0}' - a\frac{2R'^2 - RR'' + a^2}{8\sigma_k^2 R^3}b_{k0}\Big)\Big] + \ldots\Big\} \quad (6.3)$$

$$\frac{\partial^2 B_k}{\partial n^2}\bigg|_\Gamma = \frac{1}{\lambda^2}\Big\{\sigma_k^2 b_{k0} + \lambda\Big(\sigma_k^2 b_{k1} - \frac{\sigma_k a}{R}b_{k0}\Big) + \lambda^2\Big[\sigma_k^2 b_{k2} - \frac{\sigma_k a}{R}b_{k1} + \Big(\frac{a^2}{2R^2}b_{k0} - b''_{k0}\Big) + \lambda^3\Big[\sigma_k^2 b_{k3} - \frac{\sigma_k a}{R}b_{k2} + \Big(\frac{a^2}{2R^2}b_{k1} - b_{k1}''\Big) + \Big(\frac{a}{2\sigma_k R}b_{k0}'' + \frac{a^3}{8\sigma_k R^3}b_{k0}\Big)\Big] + \ldots\Big\}$$

If now (6.1) to (6.3) and the expansions in form (3.3) are substituted into system (4.20), (4.21), (5.1) and (5.2), and terms of the same order in λ are grouped together, then the system can be written out in the following form:

$$P_1\lambda + P_2\lambda^2 + P_3\lambda^3 + \ldots = 0, \qquad R_1\lambda + R_2\lambda^2 + R_3\lambda^3 + \ldots = 0$$
$$S_{m1}\lambda + S_{m2}\lambda^2 + S_{m3}\lambda^3 + \ldots = 0 \quad (m = 0, 1, 2, \ldots) \qquad (6.4)$$
$$T_{t1}\lambda + T_{t2}\lambda^2 + T_{t3}\lambda^3 + \ldots = 0 \quad (t = 1, 2, \ldots)$$

By equating to zero the coefficients of like powers of λ on the left-hand sides of (6.4), we obtain a system of recurrence relations from which one can successively find all the ψ_i, b_{mi} and c_{ti}. In the first approximation we have

$$P_1 = 2(\nu^2 - 1) \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} \frac{a}{R} c_{p0} - 4\nu(\nu - 1) \times \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} \oint_\Gamma c_{p0}(\sigma) K_{12}(\sigma, s)\, d\sigma = 0 \tag{6.5}$$

$$R_1 = 2(\nu^2 - 1) \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} c_{p0}'' - 4\nu(\nu - 1) \times \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} \oint_\Gamma c_{p0}(\sigma) K_{11}(\sigma, s)\, d\sigma = 0 \tag{6.6}$$

$$S_{m1} = -\sigma_m b_{m0} - \frac{2(-1)^{m+1}}{\sigma_m} \sum_{p=1}^{\infty} \frac{\cos^2 \gamma_p}{\gamma_p^2 - \sigma_m^2} \Big\{ (\nu - 1)\gamma_p^2 c_{p0}' + \left(1 - \nu \frac{\gamma_p^2 + \sigma_m^2}{\gamma_p^2 - \sigma_m^2}\right)(\gamma_p^2 + \sigma_m \gamma_p) c_{p0}' \Big\} = 0 \quad (m = 0, 1, 2, \ldots) \tag{6.7}$$

$$T_{t1} = -4\nu \sum_{p=1 (p \neq t)}^{\infty} \frac{\gamma_p^2 \gamma_t^2 (\cos^2 \gamma_t - \cos^2 \gamma_p)}{(\gamma_t^2 - \gamma_p^2)^2 (\gamma_t - \gamma_p)} [(\nu - 1)(\gamma_p^2 + \gamma_t^2) + 2(\nu + 1)\gamma_p \gamma_t] c_{p0} + 2\nu^2 \gamma_t^3 \left(\frac{2}{3}\cos^2 \gamma_t - 1\right) c_{t0} = 0 \quad (t = 1, 2, \ldots) \tag{6.8}$$

A more detailed investigation of system (6.8) shows that it has only the zero solution $c_{t0} \equiv 0$, if there is interest in solutions that lead to physically meaningful states of stress and strain. Then, from (6.7) we obtain: $b_{m0} \equiv 0$. In this connection, equations (6.5) and (6.6) will have been satisfied.

By equating the coefficients of λ^2, we obtain the following relations:

$$P_2 = \frac{2(\nu + 1)}{3} \left[2\nu \frac{\partial^2 \psi_0}{\partial n^2} + (\nu - 1)\left(\frac{\partial^2 \psi_0}{\partial s^2} + \frac{a}{R} \frac{\partial \psi_0}{\partial n} \right) \right]_{n=0} + 2(\nu^2 - 1) \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} \frac{a}{R} c_{p1} - 4\nu(\nu - 1) \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} \oint_\Gamma c_{p1}(\sigma) K_{12}(\sigma, s)\, d\sigma - \frac{\nu + 1}{2\mu} M_{11} = 0 \tag{6.9}$$

$$R_2 = -\frac{2}{3}(\nu + 1)\left[2\nu \frac{\partial \Delta \psi_0}{\partial n} + (\nu + 1)\left(\frac{\partial^3 \psi_0}{\partial n \partial s^2} + \frac{aR'}{R^2} \frac{\partial \psi_0}{\partial s} - \frac{a}{R} \frac{\partial^2 \psi_0}{\partial s^2} \right) \right]_{n=0} + 2(\nu^2 - 1) \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} c_{p1}'' - 4\nu(\nu - 1) \sum_{p=1}^{\infty} \frac{\sin^2 \gamma_p}{\gamma_p} \oint_\Gamma c_{p1}(\sigma) K_{11}(\sigma, s)\, d\sigma + \frac{\nu + 1}{2\mu}\left(Q_{02} + \frac{\partial G_{11}}{\partial s} \right) = 0 \tag{6.10}$$

$$S_{m2}=\frac{2(-1)^{m+1}(\nu+1)}{\sigma_m^2}\left(\frac{\partial^2\psi_0}{\partial s\partial n}-\frac{a}{R}\frac{\partial\psi_0}{\partial s}\right)_{n=0}-\sigma_m b_{m1}-$$
$$-\frac{2(-1)^{m+1}}{\sigma_m}\sum_{p=1}^{\infty}\frac{\cos^2\gamma_p}{\gamma_p^2-\sigma_m^2}\left\{(\nu-1)\gamma_p^2+\right.$$
$$\left.+\left(1-\nu\frac{\gamma_p^2+\sigma_m^2}{\gamma_p^2-\sigma_m^2}\right)(\gamma_p^2+\sigma_m\gamma_p)\right\}c_{p1}'-\frac{\sigma_m}{2\mu}T_{m1}=0\qquad(m=0,1,2,\ldots)\quad(6.11)$$

$$T_{t2}=-4\nu\sum_{p=1(p\neq t)}^{\infty}\frac{\gamma_p^2\gamma_t^2(\cos^2\gamma_t-\cos^2\gamma_p)}{(\gamma_t^2-\gamma_p^2)^2(\gamma_t-\gamma_p)}[(\nu-1)(\gamma_p^2+\gamma_t^2)+2(\nu+1)\gamma_p\gamma_t]\,c_{p1}+$$
$$+c_1+2\nu^2\gamma_t^3\left(\frac{2}{3}\cos^2\gamma_t-1\right)c_{t1}=0\qquad(t=1,2,3,\ldots)\qquad(6.12)$$

From system (6.12) we again find that $c_{t1}\equiv 0$. From (6.9), (6.10), we obtain the boundary conditions for the determination of ψ_0, which, as is easily seen, is identical with the boundary conditions in the technical theory of the bending of plates based on Kirchhoff's hypothesis. System (6.11) allows the determination of b_{m1}. By equationg coefficients of λ^3, we obtain

$$P_3=\frac{2}{3}(\nu+1)\left[2\nu\frac{\partial^2\psi_1}{\partial n^2}+(\nu-1)\left(\frac{\partial^2\psi_1}{\partial s^2}+\frac{a}{R}\frac{\partial\psi_1}{\partial n}\right)\right]_{n=0}+$$
$$+4(\nu+1)\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2}b_{k1}'+2(\nu^2-1)\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}\frac{a}{R}c_{p2}-$$
$$-4\nu(\nu-1)\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}\oint_{\Gamma}c_{p2}(\sigma)K_{12}(\sigma,s)\,d\sigma-\frac{\nu+1}{2\mu}M_{12}=0\quad(6.13)$$

$$R_3=-\frac{2}{3}(\nu+1)\left[2\nu\frac{\partial\Delta\psi_1}{\partial n}+\left(\frac{\partial^3\psi_1}{\partial n\partial s^2}+\frac{aR'}{R^2}\frac{\partial\psi_1}{\partial s}-\frac{a}{R}\frac{\partial^2\psi_1}{\partial s^2}\right)\right]_{n=0}-\quad(6.14)$$
$$-4(\nu+1)\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2}\left(-\frac{aR'}{R^2}b_{k1}+\frac{a}{R}b_{k1}'\right)+2(\nu^2-1)\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}c_{p2}''-$$
$$-4\nu(\nu-1)\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}\oint_{\Gamma}c_{p2}(\sigma)K_{11}(\sigma,s)\,d\sigma+\frac{\nu+1}{2\mu}\left(Q_{03}+\frac{\partial G_{12}}{\partial s}\right)=0$$

$$S_{m3}=\frac{2(-1)^{m+1}}{\sigma_m^3}\left\{2\nu\frac{\partial^3\psi_0}{\partial s\partial n^2}+(\nu-1)\left(\frac{\partial^3\psi_0}{\partial s^3}-\frac{aR'}{R^2}\frac{\partial\psi_0}{\partial n}+\frac{a}{R}\frac{\partial^2\psi_0}{\partial s\partial n}\right)+\right.$$
$$\left.+(\nu+1)\left[\sigma_m\left(\frac{\partial^2\psi_1}{\partial s\partial n}-\frac{a}{R}\frac{\partial\psi_1}{\partial s}\right)-\frac{a}{2R}\left(\frac{\partial^2\psi_0}{\partial s\partial n}-\frac{a}{R}\frac{\partial\psi_0}{\partial s}\right)\right]\right\}_{n=0}+$$
$$+\left[-\sigma_m b_{m2}+\frac{5a}{2R}b_{m1}\right]-\frac{2(-1)^{m+1}}{\sigma_m}\sum_{p=1}^{\infty}\frac{\cos^2\gamma_p}{\gamma_p^2-\sigma_m^2}\times$$
$$\times\left\{(\nu-1)\gamma_p^2c_{p2}'+\left(1-\nu\frac{\gamma_p^2+\sigma_m^2}{\gamma_p^2-\sigma_m^2}\right)(\gamma_p^2c_{p2}'+\gamma_p\sigma_m c_{p2}')\right\}-$$

$$- \frac{1}{2\mu}\left(\frac{\partial N_{m1}}{\partial s} + \sigma_m T_{m2} - \frac{a}{2R} T_{m1}\right) = 0 \qquad (m = 0, 1, 2, \ldots) \tag{6.15}$$

$$T_{t3} = -2(1-\nu)\frac{\sin^2\gamma_t}{\gamma_t}\left[2\nu\frac{\partial^2\psi_0}{\partial n^2} + (\nu - 1)\left(\frac{\partial^2\psi_0}{\partial s^2} + \frac{a}{R}\frac{\partial\psi_0}{\partial n}\right)\right]_{n=0} -$$

$$- 4\nu\sum_{p=1(p\neq t)}^{\infty}\frac{\gamma_p^2\gamma_t^2(\cos^2\gamma_t - \cos^2\gamma_p)}{(\gamma_t^2 - \gamma_p^2)^2(\gamma_t - \gamma_p)}[(\nu - 1)(\gamma_p^2 + \gamma_t^2) + 2(\nu + 1)\gamma_p\gamma_t]\, c_{p2} +$$

$$+ 2\nu^2\gamma_t^3\left(\frac{2}{3}\cos^2\gamma_t - 1\right)c_{t2} + \frac{\gamma_t}{2\mu}N_{t1} = 0 \qquad (t = 1, 2, 3, \ldots) \tag{6.16}$$

From system (6.16) we determine c_{t2}, which, in general, are non-zero. Then, from (6.13) and (6.14), we find the boundary conditions for ψ_1, and from system (6.15) we determine b_{m2}. This process can be continued as far as desired, when Γ and the loading is sufficiently smooth.

As can easily be seen, at each stage of the construction of the asymptotic expansions it is necessary to solve exactly the same biharmonic problem for region Ω as that arising in the technical bending theory for plates based on Kirchhoff's hypothesis. In addition, it is also necessary to solve a certain infinite system of algebraic equations, i.e. to invert a certain infinite matrix. It should be particularly emphasized that the coefficients of this system are quite well-defined numbers which are independent both of the external loading and of the boundary curve Γ of the plate. The matrix of this system can be inverted once and for ever. As a subsidiary study shows, the inversion of this matrix for physically meaningful classes of solutions is quite possible and could be realized practically by means of the method of reduction.

It can also be proved that the biharmonic problem encountered at each stage of the construction of the approximation is solvable.

7. In the assumptions made about the external loading, which were mentioned in Section 3, the coefficients B_k, which characterize the rotational state of stress, have an order of smallness in λ one unit larger than the biharmonic solution, and the coefficients C_p, which characterize the potential state of stress, have an order of smallness in λ two units larger than the biharmonic solution. Therefore, expansions (6.1) actually have the forms

$$\psi(s, n) = \psi_0(s, n) + \lambda\psi_1(s, n) + \lambda^2\psi_2(s, n) + \ldots \tag{7.1}$$

$$b_k(s) = \lambda b_{k1}(s) + \lambda^2 b_{k2}(s) + \lambda^3 b_{k3}(s) + \ldots \tag{7.2}$$
$$c_p(s) = \lambda^2 c_{p2}(s) + \lambda^3 c_{p3}(s) + \lambda^4 c_{p4}(s) + \ldots$$

By making use of formulas (2.9), for B_k and C_p we obtain the asymptotic expansions

$$B_k(s,n)=\Big\{\lambda\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)b_{k1}(s)+\lambda^2\Big[\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)\times$$
$$\times\ b_{k2}(s)+\frac{1}{2\sigma_k}\Big(-n\frac{a^2}{4R^2}-n\frac{\partial^2}{\partial s^2}+\ldots\Big)b_{k1}(s)\Big]+\ldots\Big\}\exp\frac{\sigma_k n}{\lambda}$$

$$\frac{\partial B_k}{\partial n}=\Big\{\sigma_k\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)b_{k1}(s)+\lambda\Big[\Big(-\frac{a}{2R}+n\frac{3a^2}{4R^2}-\ldots\Big)b_{k1}(s)+$$
$$+\sigma_k\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)b_{k2}(s)+$$
$$+\frac{1}{2}\Big(-n\frac{a^2}{4R^2}-n\frac{\partial^2}{\partial s^2}+\ldots\Big)b_{k1}(s)\Big]+\ldots\Big\}\exp\frac{\sigma_k n}{\lambda} \tag{7.3}$$

$$\frac{\partial^2 B_k}{\partial n^2}=\Big\{\frac{1}{\lambda}\sigma_k^2\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)b_{k1}(s)+$$
$$+\Big[2\sigma_k\Big(-\frac{a}{2R}+n\frac{3a^2}{4R^2}-\ldots\Big)b_{k1}(s)+\sigma_k^2\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)b_{k2}(s)+$$
$$+\frac{\sigma_k}{2}\Big(-n\frac{a^2}{4R^2}-n\frac{\partial^2}{\partial s^2}+\ldots\Big)b_{k1}(s)\Big]+\ldots\Big\}\exp\frac{\sigma_k n}{\lambda}$$

$$C_p(s,n)=\Big\{\lambda^2\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)c_{p2}(s)+\lambda^3\Big[\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)\times$$
$$\times\ c_{p3}(s)+\frac{1}{2\gamma_p}\Big(-n\frac{a^2}{4R^2}-n\frac{\partial^2}{\partial s^2}+\ldots\Big)c_{p2}(s)\Big]+\ldots\Big\}\exp\frac{\gamma_p n}{\lambda}$$

$$\frac{\partial C_p}{\partial n}=\Big\{\lambda\gamma_p\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)c_{p2}(s)+\lambda^2\Big[\Big(-\frac{a}{2R}+n\frac{3a^2}{4R^2}-\ldots\Big)\times$$
$$\times\ c_{p2}(s)+\gamma_p\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)c_{p3}(s)+$$
$$+\frac{1}{2}\Big(-n\frac{a^2}{4R^2}-n\frac{\partial^2}{\partial s^2}+\ldots\Big)c_{p2}(s)\Big]+\ldots\Big\}\exp\frac{\gamma_p n}{\lambda} \tag{7.4}$$

$$\frac{\partial^2 C_p}{\partial n^2}=\Big\{\gamma_p^2\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)c_{p2}(s)+$$
$$+\lambda\Big[2\gamma_p\Big(-\frac{a}{2R}+n\frac{3a^2}{4R^2}-\ldots\Big)c_{p2}(s)+\gamma_p^2\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)c_{p3}(s)+$$
$$+\frac{\gamma_p}{2}\Big(-n\frac{a^2}{4R^2}-n\frac{\partial^2}{\partial s^2}+\ldots\Big)c_{p2}(s)\Big]+\ldots\Big\}\exp\frac{\gamma_p n}{\lambda}$$

If (7.1), (7.3) and (7.4) are substituted into formulas (4.6) and (4.7), we obtain

$$\sigma_n=2\mu\lambda\Big[2\nu\frac{\partial^2\psi_0}{\partial n^2}+(\nu-1)\Big(\frac{1}{H^2}\frac{\partial^2\psi_0}{\partial s^2}+\frac{1}{H}\frac{a}{R}\frac{\partial\psi_0}{\partial n}+\frac{1}{H^3}n\frac{aR'}{R^2}\frac{\partial\psi_0}{\partial s}\Big)\Big]\zeta+$$
$$+2\mu\lambda\Big[(\nu-1)\sum_{p=1}^{\infty}s_p(\zeta)\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)c_{p2}(s)+$$
$$+\sum_{p=1}^{\infty}n_p(\zeta)\gamma_p^2\Big(1-n\frac{a}{2R}+n^2\frac{3a^2}{8R^2}-\ldots\Big)c_{p2}(s)\Big]\exp\frac{\gamma_p n}{\lambda}+$$

$$+ 2\mu\lambda^2 \left\{\left[2\nu \frac{\partial^2\psi_1}{\partial n^2} + (\nu - 1)\left(\frac{1}{H^2}\frac{\partial^2\psi_1}{\partial s^2} + \frac{1}{H}\frac{a}{R}\frac{\partial\psi_1}{\partial n} + \frac{1}{H^3} n \frac{aR'}{R^2}\frac{\partial\psi_1}{\partial s}\right)\right]\zeta + \right.$$

$$+ 2\sum_{k=0}^{\infty} \sin\sigma_k\zeta \frac{1}{H}\left[b_{k1}(s)\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)\right]_s' \exp\frac{\sigma_k n}{\lambda} +$$

$$+ (\nu - 1)\sum_{p=1}^{\infty} s_p(\zeta)\left[\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)c_{p3}(s) + \frac{1}{2\gamma_p}\left(- n\frac{a^2}{4R^2} - \right.\right.$$

$$\left.\left.- n\frac{\partial^2}{\partial s^2} + \ldots\right)c_{p2}(s)\right]\exp\frac{\gamma_p n}{\lambda} + \sum_{p=1}^{\infty} n_p(\zeta)\gamma_p^2\left[\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)c_{p3}(s) + \right.$$

$$\left. + \frac{1}{2\gamma_p}\left(- n\frac{a^2}{4R^2} - n\frac{\partial^2}{\partial s^2} + \ldots\right)c_{p2}(s)\right]\exp\frac{\gamma_p n}{\lambda} +$$

$$\left. + \sum_{p=1}^{\infty} n_p(\zeta)\gamma_p\left(- \frac{a}{R} + n\frac{3a^2}{2R^2} + \ldots\right)c_{p2}(s)\exp\frac{\gamma_p n}{\lambda}\right\} + \ldots \qquad (7.5)$$

$$\tau_{ns} = 2\mu\lambda(\nu + 1)\zeta\left(\frac{1}{H}\frac{\partial^2\psi_0}{\partial n\,\partial s} - \frac{1}{H^2}\frac{a}{R}\frac{\partial\psi_0}{\partial s}\right) + \qquad (7.6)$$

$$+ 2\mu\lambda\left\{- \sum_{k=0}^{\infty}\sigma_k \sin\sigma_k\zeta\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)b_{k1}(s)\exp\frac{\sigma_k n}{\lambda}\right\} +$$

$$+ 2\mu\lambda^2\left\{(\nu + 1)\zeta\left(\frac{1}{H}\frac{\partial^2\psi_1}{\partial n\,\partial s} - \frac{1}{H^2}\frac{a}{R}\frac{\partial\psi_1}{\partial s}\right) + \right.$$

$$+ \sum_{k=0}^{\infty}\sin\sigma_k\zeta\left[\frac{1}{H}\frac{a}{R}\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)b_{k1}(s) - \right.$$

$$- \sigma_k\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)b_{k2}(s) - \frac{1}{2}\left(- n\frac{a^2}{4R^2} - n\frac{\partial^2}{\partial s^2} + \ldots\right)b_{k1}(s) -$$

$$\left. - \left(- \frac{a}{R} + n\frac{3a^2}{2R^2} - \ldots\right)b_{k1}(s)\right]\exp\frac{\sigma_k n}{\lambda} +$$

$$\left. + \sum_{p=1}^{\infty} n_p(\zeta)\gamma_p H^{-1}\left[\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)c_{p2}(s)\right]_s' \exp\frac{\gamma_p n}{\lambda}\right\} + \ldots$$

$$\tau_{nz} = 2\mu\lambda\sum_{p=1}^{\infty} r_p(\zeta)\gamma_p\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)c_{p2}(s)\exp\frac{\gamma_p n}{\lambda} +$$

$$+ 2\mu\lambda^2\left\{\nu(1 - \zeta^2)\frac{\partial\Delta\psi_0}{\partial n} + \sum_{k=0}^{\infty}\cos\sigma_k\zeta\left[\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)b_{k1}(s)\right]_s \exp\frac{\sigma_k n}{\lambda} + \right.$$

$$+ \sum_{p=1}^{\infty} r_p(\zeta)\left[\gamma_p\left(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\right)c_{p3}(s) + \frac{1}{2}\left(- n\frac{a^2}{4R^2} - \right.\right.$$

$$\left.\left.\left. - n\frac{\partial^2}{\partial s^2} + \ldots\right)c_{p2}(s) + \left(- \frac{a}{2R} + n\frac{3a^2}{4R^2} - \ldots\right)c_{p2}(s)\right]\exp\frac{\gamma_p n}{\lambda}\right\} + \ldots \qquad (7.7)$$

$$u_n = \lambda(\nu+1)a\zeta\frac{\partial\psi_0}{\partial n} + \lambda^2\Big\{(\nu+1)a\zeta\frac{\partial\psi_1}{\partial n} +$$

$$+ a\sum_{p=1}^{\infty} n_p(\zeta)\gamma_p\Big(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\Big)c_{p2}(s)\exp\frac{\gamma_p n}{\lambda}\Big\} + \ldots \qquad (7.8)$$

$$u_s = \lambda(\nu+1)a\zeta H^{-1}\frac{\partial\psi_0}{\partial s} + \lambda^2\Big\{(\nu+1)a\zeta H^{-1}\frac{\partial\psi_1}{\partial s} -$$

$$- 2a\sum_{k=0}^{\infty}\sin\sigma_k\zeta\Big(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\Big)b_{k1}(s)\exp\frac{\sigma_k n}{\lambda}\Big\} + \ldots$$

$$w = -(\nu+1)a\psi_0 - \lambda(\nu+1)a\psi_1 + \lambda^2\Big[-(\nu+1)a\psi_2 - (\nu-1)\frac{\zeta^2}{2}a\Delta\psi_0 +$$

$$+ 2\nu a\Delta\psi_0 - a\sum_{p=1}^{\infty} q_p(\zeta)\Big(1 - n\frac{a}{2R} + n^2\frac{3a^2}{8R^2} - \ldots\Big)c_{p2}(s)\exp\frac{\gamma_p n}{\lambda}\Big] + \ldots$$

The first terms on the right-hand sides of formulas (7.8) correspond to the solution in the technical bending theory of plates. Hence it follows that the error in the determination of the displacements according to Kirchhoff's hypothesis will be of order λ compared with unity over the whole closed region occupied by the plate. This result is in agreement with the well-known results obtained on the basis of qualitative investigations [9, 10].

The state of affairs with the stresses is rather more complicated. If the behavior of the stresses in the closed cylindrical sub-region $\Omega' \times 2h$ (Fig. 4) is considered, then in the determination of the stresses here according to Kirchhoff's hypothesis there will be an error of order λ compared to unity.* However, this result is valid namely for all sub-regions $\Omega' \times 2h$ lying wholly within $\Omega \times 2h$. In fact, on the boundary Ω, where $n = 0$, we will have

$$\sigma_n = 2\mu\lambda\zeta\Big[2\nu\frac{\partial^2\psi_0}{\partial n^2} + (\nu-1)\Big(\frac{\partial^2\psi_0}{\partial s^2} + \frac{a}{R}\frac{\partial\psi_0}{\partial n}\Big)\Big]_{n=0} +$$

$$+ 2\mu\lambda\sum_{p=1}^{\infty}[(\nu-1)s_p(\zeta) + \gamma_p^2 n_p(\zeta)]c_{p2}(s) +$$

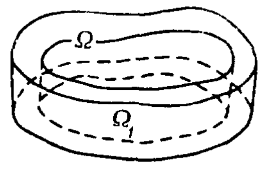

Fig. 4.

$$+ 2\mu\lambda^2\Big\{\Big[2\nu\frac{\partial^2\psi_1}{\partial n^2} + (\nu-1)\Big(\frac{\partial^2\psi_1}{\partial s^2} + \frac{a}{R}\frac{\partial\psi_1}{\partial n}\Big)\Big]\zeta + 2\sum_{k=0}^{\infty}\sin\sigma_k\zeta b'_{k1}(s) +$$

* In the calculation of the degree of error, no account has been taken, of course, of terms which have exponential order of decay relative to λ, i.e. of terms containing $\exp\sigma_k n/\lambda$ and $\exp\gamma_p n/\lambda$.

$$+\sum_{p=1}^{\infty}[(\nu-1)s_p(\zeta)+\gamma_p{}^2 n_p(\zeta)]\,c_{p3}(s)-\frac{a}{R}\sum_{p=1}^{\infty}\gamma_p n_p(\zeta)\,c_{p2}(s)\Big\}+\ldots \qquad (7.9)$$

$$\tau_{ns}=2\mu\lambda(\nu+1)\,\zeta\left(\frac{\partial^2\psi_0}{\partial n\,\partial s}-\frac{a}{R}\frac{\partial\psi_0}{\partial s}\right)-2\mu\lambda\sum_{k=0}^{\infty}\sigma_k\sin\sigma_k\zeta b_{k1}(s)+$$
$$+2\mu\lambda^2\left\{(\nu+1)\,\zeta\left(\frac{\partial^2\psi_1}{\partial n\,\partial s}-\frac{a}{R}\frac{\partial\psi_1}{\partial s}\right)+\right.$$
$$\left.+\sum_{k=0}^{\infty}\sin\sigma_k\zeta\left[\frac{2a}{R}\,b_{k1}(s)-\sigma_k b_{k2}(s)\right]+\sum_{p=1}^{\infty}n_p(\zeta)\,\gamma_p c_{p2}'(s)\right\}+\ldots \qquad (7.10)$$

$$\tau_{nz}=2\mu\lambda\sum_{p=1}^{\infty}r_p(\zeta)\,\gamma_p c_{p2}(s)+2\mu\lambda^2\left\{\nu(1-\zeta^2)\left.\frac{\partial\Delta\psi_0}{\partial n}\right|_{n=0}+\right.$$
$$\left.+\sum_{k=0}^{\infty}\cos\sigma_k\zeta b_{k1}'(s)+\sum_{p=1}^{\infty}r_p(\zeta)\left[\gamma_p c_{p3}(s)-\frac{a}{2R}\,c_{p2}(s)\right]\right\}+\ldots \qquad (7.11)$$

The first terms on the right-hand sides of (7.9) and (7.10) correspond to the solution of the technical bencing theory for plates. The remaining terms will be supplementary to the solution of the Kirchhoff theory. The formulas show that among them there are terms of the same order in λ as in the solution for the technical bending theory. In other words, the error in the application of Kirchhoff's hypothesis will have zero order in λ relative to unity as $\lambda \to 0$. If expression (7.11) for τ_{nz} is considered, then as $\lambda \to 0$ the corrections to the terms given by the Kirchhoff hypothesis begin to play an important part, and here the error in the Kirchhoff hypothesis turns out to be arbitrarily large.

These conclusions, in our opinion, are of great importance. They show, for example, that it is necessary to exercise caution in estimating the state of stress near the boundary of the plate on the basis of Kirchhoff's hypothesis. The last feature is of great practical importance because during the bending of plates the greatest stresses usually occur at the boundary of cutouts and from these stresses one can calculate the stress-concentration factor. The method developed here makes it possible to study the question of the accuracy of the Kirchhoff theory in the calculation of stress concentrations.

Further, let us note the following fact: we will assume that on each section s = const we have a statically self-equilibrating loading. In this case, as can be easily seen, $\psi_0 = 0$, and ψ_1 is non-zero in general. This indicates that in the present case, in spite of the fact that the loading on each section is self-equilibrating, there will be a state of stress penetrating without decay into the depth of the plate. However, it will be of a higher order. The stress concentration in this case, it is clear, cannot be found from the technical bending theory for plates.

BIBLIOGRAPHY

1. Friedrichs, K.O. and Dressler, R.F., A boundary-layer theory for elastic plates. *Comm. Pure Appl. Math.*, Vol. 14, No. 1, 1961.

2. Green, A.E., On the linear theory of thin elastic shells. *Proc. Roy. Soc.*, A266, No. 1325, pp. 143-160, 1962.

3. Reiss, E.L., Symmetric bending of thick circular plates. *J. Soc. Industr. Appl. Math.*, Vol. 10, No. 4, 1962.

4. Gol'denveizer, A.L., Postroenie priblizhennoi teorii izgiba plastinki metodom asimptoticheskogo integrirovaniia uravnenii teorii uprugosti (Derivation of an approximate theory of bending of plates by the method of asymptotic integration of the equations of the theory of elasticity). *PMM* Vol. 26, No. 4, 1962.

5. Lur'e, A.I., *Three-dimensional Problems in the Theory of Elasticity.* Gostekhizdat, 1955. English translation published by John Wiley and and Sons.

6. Lur'e, A.I., K teorii tolstykh plit (Theory of thick plates). *PMM* Vol. 6, Nos. 2-3, 1942.

7. Vishik, M.I. and Liusternik, L.A., Reguliarnoe vyrozhdenie i pogranichnyi sloi dlia lineinykh differentsial'nykh uravnenii s malym parametrom (Regular degeneration and boundary layer for linear differential equations with a small parameter). *Uspekhi matem. nauk*, Vol. 12, No. 5 (77), 1957.

8. Leibenzon, L.S., Kurs teorii uprugosti (Course in the Theory of Elasticity). Gostekhizdat, 1947.

9. Novozhilov, V.V. and Finkel'shtein, R.M., O pogreshnostiakh gipotez Kirkhgofa v teorii obolochek (On the errors of Kirchhoff's hypothesis in the theory of shells). *PMM* Vol. 7, 1943.

10. Mushtari, Kh.M., Ob oblasti primenimosti priblizhennoi teorii obolochek Kirkhgofa-Liava (On the area of application of the approximate Kirchhoff-Love theory of shells). *PMM* Vol. 11, No. 5, 1947.

Translated by D. McV.